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LETTER TO THE EDITOR

Superconductivity from repulsion: a variational view

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Abstract. This letter presents a new class of variational wavefunctions for Fermi systems in any dimension. These wavefunctions introduce correlations between Cooper pairs in different momentum states and the relevant correlations can be computed analytically. At half filling we have a ground state with critical superconducting correlations, that causes negligible increase of the kinetic energy. We find large enhancements in a Cooper-pair correlation function caused purely by the interplay between the uncertainty principle, repulsion and the proximity of half filling. This is surprising since there is no accompanying signature in usual charge and spin response functions, and typifies a novel kind of many-body cooperative behaviour.

Introduction

Variational wavefunctions have played an important part in our understanding of several phenomena in quantum many-body systems. The role of the BCS [1] wavefunction in superconductivity, Jastrow-type functions for superfluidity [2], the Laughlin function in the fractional quantum Hall problem [3], and the Gutzwiller wavefunction [4] in strongly correlated Fermi systems have been recognized to be of fundamental importance.

This letter presents a new variational wavefunction and some variants of it, that promise to be of interest in the topical problem of superconductivity arising from repulsion. We have recently argued on the basis of certain inequalities [5], that projecting out s-wave Cooper pairs in a Fermi system *on a lattice* would lead to enhanced extended s-wave-type pairing fluctuations near half filling. These are expected to lead to superconductivity at precisely half filling for a class of Hubbard-type models. In brief, the argument involves the recognition that s-wave and extended s-wave pairing are canonically conjugate in the sense of the uncertainty principle, which ultimately drives the instability towards superconductivity near half filling. This conjugacy arises since the s-wave and the extended s-wave Cooper order parameters B, A (defined below) satisfy the commutation relation $[B, A^\dagger] = 2T$, where the kinetic energy operator T plays a benign role, similar to that of the number operator $\hat{N}$ as in other familiar contexts, such as the conductivity sum rule on the lattice. From the uncertainty principle, it follows [5] that the fluctuation $\langle A^\dagger A + AA^\dagger\rangle$ is bounded from below by $4\langle T\rangle^2/\langle B^\dagger B + BB^\dagger\rangle$. The special role played by half filling is due to the fact that the suppression of s-wave fluctuations is possible to a very high degree near half filling, the commutation relation $[B, B^\dagger] = L - \hat{N}$ in a sector $\langle\hat{N}\rangle \sim \mathcal{L}$, permits both $\langle B^\dagger B\rangle$ and $\langle BB^\dagger\rangle$ to be simultaneously small. This 'squeezing' results in large enhancements in the conjugate variables, remarkably enough, without an appreciable energy cost. This effect has been termed *order by projection*.

While the arguments in [5] provide a novel direction, the absence of explicit solutions or of good wavefunctions has proved to be a hindrance in arriving at a thorough understanding of the physics of these models. In this work we provide a wavefunction which is tractable

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enough so that calculations of all relevant expectation values can be performed analytically, which by itself is rare in many-body systems. Further, the wavefunction catches the essence of the enhancements mentioned above, and at half filling is argued to be exact. While we know of no materials to which these models apply, we make venture to present these models and wavefunctions, since they address and provide what we believe to be amongst the first positive results [6] for the important theoretical question of the possibility of superconductivity without explicit attractive interactions—in fact arising out of pure repulsion.

The model and the variational wavefunction

Let us write the Hamiltonian of the model considered in d dimensions as

$$H = T + U_s B^\dagger B. \tag{1}$$

Here $B = \sum b_k$ is the s-wave Cooper-pair operator, and $b_k = c_{-k\downarrow} c_{k\uparrow}$ the pair operators, the kinetic energy $T = \sum \epsilon_k n_{k\sigma}$, with the band dispersion $\epsilon_k = -2\sum_{\alpha=1}^{d} \cos(k_\alpha)$, although later we consider a slightly more general form of the band dispersion in two dimensions with orthorhombic distortion. The U_s term discourages on-site s-wave correlations, and in fact projects out s-wave order. In order to get a feeling for its effect, consider the estimate of the energy in a BCS state $|\Psi_{BCS}\rangle = \prod(u_k + v_k b_k^\dagger)|\text{vac}\rangle$. Provided $\sum u_k v_k = 0$, we find $E_{BCS} = \sum[2(\epsilon_k - \mu)v_k^2 + U_s v_k^4]$, subject to $N = 2\sum v_k^2$. The role of U_s is clearly to flatten out v_k from its step function behaviour in the normal state, and hence a superconducting state with true long-range order (LRO) and an extensive energy shift arises, at least as a local variational minimum. In [5], a more general model is introduced including the above term as well as the Hubbard U term. The general results of [5] imply for these models, that away from half filling, the ground state energy density† is unshifted by the U_s term, while the extended s-wave correlations get a large enhancement at the expense of suppressing s-wave correlations. By continuity in density, we expect superconductivity at half filling. In this work we ignore the Hubbard U term for tractability, and explore the other aspects within the model of equation (1), in particular the origin of the enhancements.

The extended s-wave operator $A = -2\sum \epsilon_k b_k$ satisfies the commutation relations $[B, T] = -A$ and $[B, A^\dagger] = 2T$. The main variational wavefunction proposed here is written in terms of the free Fermi wavefunction $|\Phi\rangle = \prod_{|k|<k_f} b_k^\dagger|\text{vac}\rangle$ as

$$|\Psi_\theta\rangle = \exp\left(-\frac{\theta}{\mathcal{L}} S\right)|\Phi\rangle \tag{2}$$

where the number of lattice sites is $\mathcal{L}$. The prefactor generates several Cooper particle pairs and Cooper hole pairs in the Fermi gas. To motivate this wavefunction, note that the anti-Hermitian operator $S = \sum(\epsilon_{k_1} - \epsilon_{k_2}) b_{k_1}^\dagger b_{k_2}$ can be viewed as the commutator $\frac{1}{2}[T, B^\dagger B]$, and hence the wavefunction may be viewed as a 'rotation' about a direction orthogonal to the kinetic and potential energies. Such a strategy is familiar e.g. from diagonalizing a sum over two Pauli matrices, as well as quadratic forms in bosons. A similar rotation also gives the exact answer in a further simplified model [7].

† The term 'energy density' is used to denote $\lim_{\mathcal{L}->\infty} \frac{E_0}{\mathcal{L}}$. We use the term 'energy per site', on the other hand, to denote $\frac{E_0}{\mathcal{L}}$ preparatory to taking the thermodynamic limit, and display only θ-dependent corrections such as $\frac{1}{\mathcal{L}^\alpha} g(\theta)$. We omit displaying other finite-size corrections that are undoubtedly present but uninteresting, being θ-independent. We also occasionally omit displaying θ-dependent but smaller corrections than the ones displayed.

Calculation of expectation values

We now turn to the calculation of expectation values $\langle Q\rangle_\theta \equiv \langle\Psi_\theta|Q|\Psi_\theta\rangle$ of various operators Q in the above state. We begin by evaluating the derivative of the kinetic energy $\langle T\rangle_\theta$,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\langle T\rangle_\theta = \frac{1}{\mathcal{L}}[\langle A^\dagger A\rangle_\theta - 2\langle C^\dagger B + B^\dagger C\rangle_\theta] \equiv 4\alpha(\theta) - 2\nu(\theta) \tag{3}$$

where $C = \sum \epsilon_k^2 b_k$, thus defining the functions α, ν. We immediately note a feature of considerable importance, namely that unless one of the correlation functions in the right-hand side of equation (3) possesses explicit LRO, the change in the (extensive) kinetic energy is *not extensive*, and may be neglected in the thermodynamic limit. We make repeated use of this insight later. We also note that this includes the case where one of the correlators possesses quasi-LRO, i.e. $\sim\mathcal{L}^{1+\zeta}$, with $\zeta < 1$, indeed we find later that $\zeta = \frac{1}{2}$ does occur for the case of greatest interest, namely the half-filled limit. We also need the expectation of operators $T_n \equiv \sum \epsilon_k^n \phi_k$, with $\phi_k = n_{k\uparrow} + n_{-k\downarrow} - 1$, so that $T_1 = T$. The equations are most compactly written in terms of the operators $I_n = \sum \epsilon_k^n b_k$ such that $I_0 = B$, $I_1 = -\frac{1}{2}A$, $I_2 = C$, and their correlators $\Gamma_{n,m}(\theta) \equiv \frac{1}{\mathcal{L}}\langle I_n^\dagger I_m + I_m^\dagger I_n\rangle_\theta$ as

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\langle T_n\rangle_\theta = 2[\Gamma_{n,1}(\theta) - \Gamma_{n,0}(\theta)]. \tag{4}$$

We note that in view of the above definition, LRO corresponds to $\Gamma \sim \mathrm{O}(\mathcal{L})$, whereas in the absence of true LRO, $\Gamma \sim \mathrm{o}(\mathcal{L})$, and hence here the correlators $\langle T_n\rangle_\theta$ are insensitive to the rotation, at least to leading order in $\mathcal{L}$.

The correlators $\Gamma_{n,m}$ satisfy exact equations

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\Gamma_{n,m}(\theta) = \frac{1}{\mathcal{L}^2}\langle[T_{n+1}I_0^\dagger I_m + I_n^\dagger I_0 T_{m+1} - I_n^\dagger I_1 T_m - T_n I_1^\dagger I_m + \mathrm{h.c.}]\rangle_\theta, \tag{5}$$

with initial conditions obtained by taking the expectation value in the free Fermi ground state: they read $\Gamma_{n,m}(0) = 2[\epsilon_k^{n+m} f_k]_k$, with f_k the usual (noninteracting) Fermi function and $[g_k]_k \equiv \frac{1}{\mathcal{L}}\sum_k g_k$.

The equations are handled assuming a fundamental factorization that arises in the thermodynamic limit. This factorization of Hermitian global operators (i.e. sums over all sites of local operators) and may be expressed as the statement that for any two such Q_j with nonzero averages, we have $\langle Q_1 Q_2\rangle_\theta/[\langle Q_1\rangle_\theta\langle Q_2\rangle_\theta] \sim \mathrm{o}(\mathcal{L})$. This is also equivalent to the statement that the connected part of the correlator is subleading in powers of $\mathcal{L}$. The case needed by us corresponds to $Q_1 \sim \mathcal{L}\Gamma_{n,m}$ and $Q_2 \sim T_n$. If this is assumed then we immediately see that equations (5) factor out into linear equations

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\Gamma_{n,m} = \mu_{n+1}\Gamma_{0,m} + \mu_{m+1}\Gamma_{0,n} - \mu_m\Gamma_{1,n} - \mu_n\Gamma_{1,m} \tag{6}$$

with coefficients $\mu_n \equiv [\epsilon_k^n(2f_k - 1)]_k$ obtainable numerically, together with the initial conditions $\Gamma_{n,m}(0) = 2[\epsilon_k^{n+m} f_k]_k$.

We have checked this assumption to low orders by evaluating the correlation functions $\Gamma_{n,m}$ for the most relevant cases $(n, m) = (0, 0), (1, 1)$ out to $\mathrm{O}(\theta^5)$. The calculation proceeds by expanding the expectation value as a nested commutator with S, which is evaluated by Wick's theorem for ground state correlations, leading to $n!$ contractions that are further multiplied by powers of ϵ. This results in sums over k of algebraic functions of the band energies and Fermi functions. The series agrees with the one generated from the above factorization procedure exactly to that order, and provides nontrivial support for it.

We now consider the correlation functions of interest, and begin by denoting $\beta(\theta) = \frac{1}{2}\Gamma_{0,0}$, $\alpha(\theta) = \frac{1}{2}\Gamma_{1,1}$, $\gamma(\theta) = \Gamma_{1,0}$ and finally $\nu(\theta) = \Gamma_{2,0}$. The equations of immediate concern are

then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\alpha &= -2\mu_1\alpha + \mu_2\gamma \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\beta &= 2\mu_1\beta + (1-\rho)\gamma \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\gamma &= 2(1-\rho)\alpha + 2\mu_2\beta.\end{aligned} \tag{7}$$

We also note, for later use, that $\frac{\mathrm{d}}{\mathrm{d}\theta}\nu = \mu_1\nu + 2\mu_3\beta + (1-\rho)\Gamma_{1,2} - \mu_2\gamma$. We denote the particle density as ρ such that $\rho \leqslant 1$. The initial conditions are $\beta(0) = \frac{1}{2}\rho$, $\alpha(0) = \nu(0) = d + \frac{1}{2}\mu_2$, $\gamma(0) = \mu_1$. We now examine the behaviour of these coupled equations. Let us note that $\mu_0 = (\rho - 1)$, further, μ_1 is negative and nonvanishing near half filling, whereas μ_2 also negative, is very small $\sim \mathrm{O}((1-\rho)^3)$ near half filling. In fact, for all even n we must have $\mu_n \to 0$ as we approach half filling, due to particle hole symmetry. If we neglect the cross couplings, then β falls off as θ increases from zero while α increases.

A neat way to solve these equations is suggested by inspection: we observe that in effect equations (7) are generated by a linear relation $\frac{\mathrm{d}}{\mathrm{d}\theta}I_n = \mu_{n+1}I_0 - \mu_n I_1$ and the hermitean conjugates of these. The θ-dependence of all the operators are thus related to the fundamental ones I_0, I_1, and thus we find linear combinations that have simple evolution: $Q_\pm = -\mu_0 I_1 + (\mu_1 \pm \lambda_0)I_0$ where $\lambda_0 = \sqrt{\mu_1^2 - \mu_0\mu_2}$. These obey uncoupled equations $\frac{\mathrm{d}}{\mathrm{d}\theta}Q_\pm = \pm\lambda_0 Q_\pm$.

We readily invert and find with $\xi_\pm = (1 \pm \frac{\mu_1}{\lambda_0})$, $I_0 = \frac{1}{2\lambda_0}\{Q_+ - Q_-\}$ and $I_1 = \frac{1}{2(1-\rho)}\{\xi_- Q_+ + \xi_+ Q_-\}$. The four basic correlators $F_{\sigma_1,\sigma_2}(\theta) = \langle Q^\dagger_{\sigma_1} Q_{\sigma_2}\rangle_\theta$ can, therefore, be found simply: $F_{\sigma_1,\sigma_2}(\theta) = F_{\sigma_1,\sigma_2}(0)\exp(\lambda_0\theta(\sigma_1+\sigma_2))$. The initial conditions are given as

$$F_{\sigma_1,\sigma_2}(0) = (1-\rho)^2\alpha(0) + (1-\rho)\left\{\mu_1 + \frac{\lambda_0}{2}(\sigma_1+\sigma_2)\right\}\gamma(0) + (\mu_1+\sigma_1\lambda_0)(\mu_1+\sigma_2\lambda_0)\beta(0). \tag{8}$$

We thus find the expectation values:

$$\begin{aligned}\alpha(\theta) &= \frac{1}{4(1-\rho)^2}[\xi_-^2 F_{1,1}(0)\exp(2\lambda_0\theta) + \xi_+^2 F_{-1,-1}(0)\exp(-2\lambda_0\theta) + 2\xi_+\xi_- F_{1,-1}(0)] \\ \beta(\theta) &= \frac{1}{4\lambda_0^2}[F_{1,1}(0)\exp(2\lambda_0\theta) + F_{-1,-1}(0)\exp(-2\lambda_0\theta) - 2F_{1,-1}(0)] \\ \gamma(\theta) &= \frac{1}{2\lambda_0(1-\rho)}\left[\xi_- F_{1,1}(0)\exp(2\lambda_0\theta) - \xi_+ F_{-1,-1}(0)\exp(-2\lambda_0\theta) + 2\left(\frac{\mu_1}{\lambda_0}\right)F_{1,-1}(0)\right].\end{aligned} \tag{9}$$

These, together with equation (3), provide the solution to the variational problem since the variational energy density (see footnote p 346) is $\mu_1 + U_s\beta(\theta)$, as well as that of computing the correlations.

In figure 1 we show the behaviour of the three functions α, β, γ for the case of two dimensions at a typical value of the density $\rho = 0.75$. Note that β goes through a very shallow minimum with a value $\beta^*(\rho) \equiv \beta(\theta^*) \sim 0$, and γ also seems to go through zero at nearly the same value of the minimizing $\theta = \theta^*$. From equations (7), β^* would actually be zero if γ vanishes at exactly θ^*. To investigate this further, in figure 2 we plot the minimum value $\beta^*(\rho)$ as a function of density in two dimensions, and the inset shows a similar plot for one dimension. We see that for general densities, although β^* is impressively small, it is nonzero. From the general arguments of [5] we know that the exact ground state energy density is unshifted from the noninteracting value, corresponding to $\beta_{exact} = 0$, and so our wavefunction gives a very good approximation to the true energy for most densities, provided we restrict to $U_s \sim |\mu_1|$.

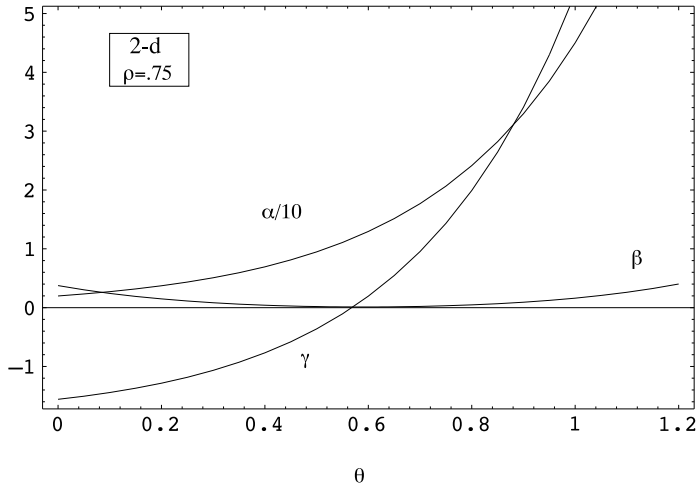

Figure 1. The three correlations functions at $\rho = 0.75$ in two dimensions versus θ.

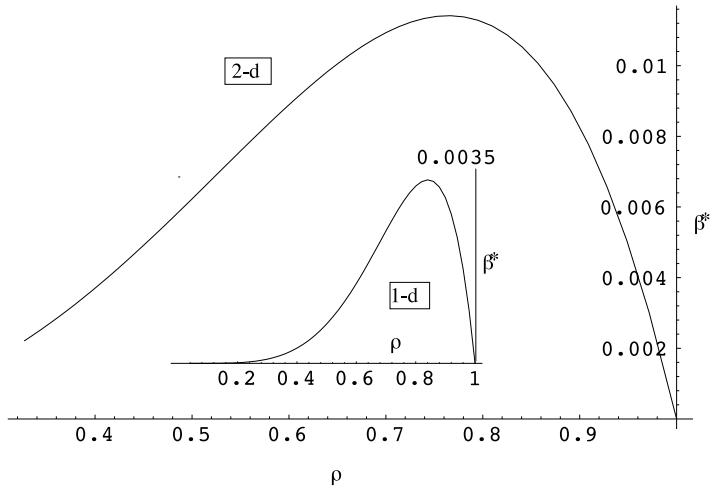

Figure 2. The minimum value of β^* versus density.

In figure 3 we show $\alpha^*(\rho)/\alpha(0)$, this is enhanced greatly, by an order of magnitude as far as 15% away from half filling. The lower bound $\alpha_{lb} = \frac{1}{2}(\frac{\mu_1^2}{(1-\rho)} + \mu_2)$ obtained in [5], also has a similar behaviour close to half filling.

Solution at half filling

The discussion of the solution above indicates that for half filling, i.e. $\rho = 1$, the wavefunction discussed here could be exact, since β^* goes to zero here. Our analysis of the orders of magnitude of the correlators was based, so far, on the implicit assumption that $\theta \sim$ O(1). From the solution it can be shown that for $\rho < 1$, the minimizing $\theta^* \sim -\log(1-\rho)$, and hence at $\rho = 1$, we must work with greater precision to see if the assumptions made earlier are consistent. First we assume that equations (7) are still valid at half filling, requiring

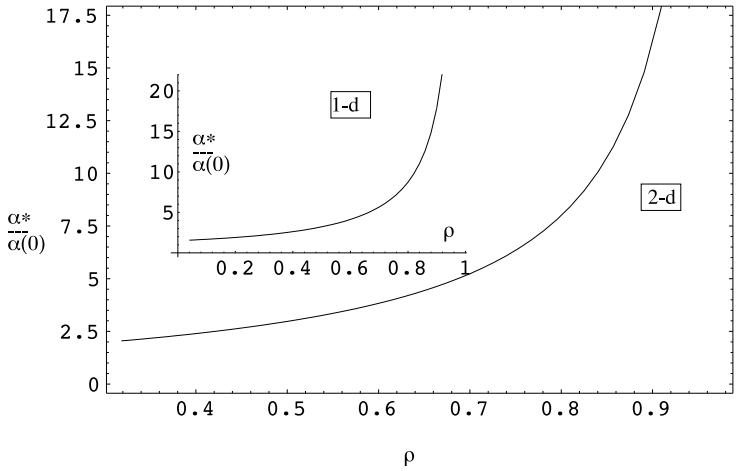

Figure 3. The enhancement factor of the extended s-wave correlator $\alpha^*/\alpha(0)$ versus density in two dimensions, and in one dimension (inset).

that $\lim_{\mathcal{L}\to\infty}\langle T_n\rangle_\theta/\mathcal{L}$ is unchanged from its noninteracting value μ_n. We verify, at the end, that this is still true and hence obtain a self-consistent argument. With the assumption we set $\mu_{2n} \to 0$ and hence the equations decouple and are integrated immediately giving $\alpha(\theta) = \alpha(0)\exp(-2\mu_1\theta)$ and $\beta(\theta) = \beta(0)\exp(2\mu_1\theta)$. We also integrate the equation for ν and find $\nu(\theta) = g_1\exp(\mu_1\theta) + g_2\exp(2\mu_1\theta)$. The constants g_1, g_2 involve the initial conditions, but are not important since we see that for large (positive) θ we can safely neglect $\nu(\theta)$ (recall that $\mu_1 < 0$). We substitute for α into equation (3) and integrate over θ, to find the energy per site (see footnote p 346)

$$e(\theta) = e_{non} + \frac{U_s}{2}\exp(2\mu_1\theta) + \frac{2\alpha(0)}{|\mu_1|\mathcal{L}}\exp(-2\mu_1\theta) + (\text{negligible terms}) \quad (10)$$

where $e_{non} = \mu_1$ is the ground state energy density of the free Fermi gas, and the negligible terms are of the type $\frac{1}{\mathcal{L}}\{1, \exp(\mu_1\theta), \exp(2\mu_1\theta)\}$ etc. We see that minimizing equation (10) w.r.t. θ pushes up the energy as

$$e(\theta^*) = e_{non} + 2\frac{\sqrt{U_s\alpha(0)}}{\sqrt{|\mu_1|\mathcal{L}}} + (\text{negligible terms}) \quad (11)$$

where $\theta^* = 1/(4|\mu_1|)\ \log(U_s|\mu_1|\mathcal{L}/4\alpha(0))$. The energy density continues to be independent of U_s. Thus, the growth of θ stops on the scale of $\log(1/\mathcal{L})$, and we find $\alpha^* = \frac{1}{2}\sqrt{\alpha(0)U_s\mathcal{L}|\mu_1|}$. Since α^* does not have true LRO, our original assumption is validated, and hence we have a consistent solution. The solution at half filling thus has quasi-LRO since $\langle A^\dagger A\rangle \sim \mathcal{L}^{3/2}$, and is superconducting in the sense of having power law order, as e.g. in a two-dimensional superconductor at finite temperatures.

The d-wave superconductor

We now ask the question whether this class of wavefunctions works for d-wave superconducting correlations as well in two dimensions. This case is clearly of topical interest in the context of the two-dimensional high T_c cuprates. There is one possibility inherent in the discussion of [5], wherein we take $\hat{B} = \sum \eta_k b_k$, with $\eta_k = \text{sign}(w_k)$, and $w_k = \cos(k_x) - \cos(k_y)$. This choice of the relative phases of the k-state Cooper pairs preserves the important commutation relation

$[\hat{B}, \hat{B}^\dagger] = \mathcal{L} - \hat{N}$ (where $\hat{N}$ is the number operator), since $\eta_k^2 = 1$, and all the arguments of [5] can be taken over, exactly as for the case of extended s-wave pairing discussed above. The conjugate variable that is enhanced now is $\hat{A} = -2\sum \epsilon_k \eta_k b_k$, and the two particle wavefunction has the signs that are usually associated with d-wave pairing. The 'gap' within a weak coupling approach in this case may be taken to be $\Delta = \Delta_0 \epsilon_k \eta_k$, and thus has jumps in k space where one expects nodes. Of course, this may not be a fatal difficulty since there is no clear connection between the ground state obtained within the present framework, and the low lying excitation spectrum within a BCS framework.

Within the variational framework, we can admit d-wave pairing provided the lattice has a nonzero orthorhombic distortion, i.e. the band dispersion is asymmetric in x and y directions. We can model this by setting $\epsilon_k = -2(1-\epsilon_0)\cos(k_x) - 2(1+\epsilon_0)\cos(k_y)$ with ϵ_0 a measure of the orthorhombicity. We choose a variational wavefunction exactly as in equation (2), with $S_{DW} = \sum (w_{k_1} - w_{k_2}) b_{k_1}^\dagger b_{k_2}$ and define the d-wave pairing operator $D = \sum w_k b_k$. With this we can essentially borrow all the results of the previous calculation provided we replace $\mu_n \to \nu_n$, where $\nu_n = \frac{1}{\mathcal{L}} \langle \hat{T}_n \rangle_0$ and $\hat{T}_n = \sum w_k^n \phi_k$. The Fermi ground state has the symmetry of the ϵ_k and hence we see the need for the distortion, for in its absence, $\nu_n = 0$ for all $n \geqslant 1$. We must also replace the initial conditions appropriately, thus $\Delta(\theta) = \frac{1}{\mathcal{L}} \langle D^\dagger D \rangle_\theta$, and $\Delta(0) = [w_k^2 f_k]_k$. We need to examine the reality of $\hat{\lambda}_0 = \sqrt{\nu_1^2 + (1-\rho)\nu_2}$. In case $\hat{\lambda}_0$ is imaginary, the nature of solution changes drastically, and we have a limited enhancement, since the functions now oscillate as trigonometric functions, as θ varies. If $\hat{\lambda}_0$ is real, we can repeat everything said before and qualitatively find the same answers as for the extended s-wave case. The numerics require a knowledge of the elements ν_n and we will not discuss them further here. At half filling the state is similarly obtained, and the energy is different from that of the s-wave case only in terms of $O(1/\sqrt{\mathcal{L}})$, it is higher in general since ν_n are usually smaller than the μ_n for $n \geqslant 1$. It must be remarked that in this case the correlation function α is also enhanced similarly to Δ, since these correlations get coupled. The correct symmetry of the order parameter is then of a mixed s and d type, rather than pure s or d [8], and indeed one should, in principle, optimize the form of the S operator by taking a suitable mixture of s and d functions to get the best possible energy.

Summary and discussion

In summary, we have presented variational wavefunctions that seem interesting on several counts. In many regards, we are tempted to say that the wavefunctions are at least as interesting as the models, if not more. First, the expectation values are computable analytically and throw light on an interesting and novel relationship, namely the enhancement in certain correlations at the expense of on site s-wave correlations. At half filling we have a state that is degenerate with the free Fermi gas in the energy density, and yet has critical superconducting correlations. Is this state the exact ground state of the model in equation (1)? The coincidence of the *energy density* with the obvious lower bound e_{non}, is not a clinching argument by itself, particularly in view of the main message of our work. The nature of the correlations, on the other hand, being of the form expected from the uncertainty principle arguments, provide a significant positive indicator. We feel that certain results found here are likely to be true, such as $\alpha^* \propto \mathcal{L}^{1/2}$, and the form of energy corrections as in equation (11), without however, being able to prove them in a rigorous fashion. Of the values of the coefficients of the leading behaviour, it is less easy to be certain.

The critical behaviour of the Cooper correlation function $\langle A^\dagger A \rangle$ is of particular interest. It behaves as $\mathcal{L}^{\frac{3}{2}}$ at $\rho = 1$, and is $\propto \frac{\mathcal{L}}{1-\rho}$ otherwise, suggesting that the local density $\alpha(r)$

of the A operator has correlations $\langle \alpha^{\dagger}(r)\alpha(0)\rangle \sim \frac{1}{|r|^{d/2}} h(r/\xi(\rho))$, with a density-dependent correlation length $\xi(\rho) \sim \frac{1}{(1-\rho)^{2/d}}$, that diverges at half filling. Such a crossover is similar to that in the behaviour of the spin correlations in the Hubbard model in one dimension near half filling [9]. Note that the behaviour of this function for the free Fermi gas is $\sim \frac{1}{r^{2d}}$, and hence the correlations in this case are considerably shorter ranged than ours.

We conclude by mentioning another nontrivial implication of the ideas discussed above, but going beyond the context of the models considered here. We have seen that the Fermi gas on a lattice has the possibility of novel large enhancements in the ground state correlation functions of Cooper pairs, produced by repulsive terms of certain kinds. These are not reflected in either the total energy (and hence the compressibility) or the momentum distribution function. The latter can readily be seen to be as sharp as in the ideal Fermi case. Similarly one can check that the spin correlations are unaffected. The usual Fermi liquid enhancements in spin and charge response that herald incipient instabilities are totally missing in these systems. As a result, any bosonic excitations available, including phonons, would find the electronic system much more prone to superconductivity of the usual sort than expected on grounds of the missing Fermi liquid enhancements, and presumably be accompanied by enhanced T_c. Estimates of such effects must await information on the spectral weight in the dynamical Cooper susceptibility.

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